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Self-similar, finite-time collapse of the straight vortex-filament dodecapole

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This article is a tribute to the late Professor Rich Pelz who suddendly passed away last September. Rich was one of my best friends on the globe for a long period of time. Our friendship began at Los Alamos in New Mexico one day in 1990. He visited the Center for Nonlinear Studies where I worked as a post-doc, and gave a seminar maybe on the parallel computation of turbulence. Soon after his seminar, I remember vividly, we began chatting on the Crow instability along a vortex tube and other thing. Since then we had many many opportunities to talk, to exchange ideas, to eat and drink together in the world.

Our mutual interest was in formation of singularity for the fluid equations, in particular, in relation with the vortex motion. The following is an excerpt from the manuscript we were preparing for publication, which Kimura modified for this volume.

The vortex dodecapole [1], [2] [3], [4], the superposition of three equal-strength, orthogonal, vortex quadrupoles, is an intriguing though specific candidate initial condition for a finite-time singularity (FTS) in ideal hydrodynamics.

In this paper, we shall examine perhaps the simplest model of the vortex dodecapole in which we replace the vortex tubes with straight vortex filaments of infinitesimal thickness. The filament dodecapole, which is shown in figure , has three orthogonal vortex quadrupoles parallel to the x, y and z axes, and let us call these quadrupoles x-, y- and z-quadrupole, respectively.

First we locate a representation point at the intersection of one of the z-quadrupole with the plane of $z = 0$ in the first quadrant of the xy-plane, and call it $\mathbf{P} = (x_0, y_0, 0)$. The induced velocity vector at $\mathbf{P}$ by the other filaments lies on the xy-plane, whose components are;

$$u = \dot{x}_0 = \underbrace{+\frac{1}{2y_0} - \frac{y_0}{2(x_0^2+y_0^2)}}_{\text{from z-quadrupole}} \underbrace{-\frac{2x_0}{(y_0-x_0)^2+x_0^2} + \frac{2x_0}{(y_0+x_0)^2+x_0^2}}_{\text{from y-quadrupole}} \tag{1}$$

$$v = \dot{y}_0 = \underbrace{-\frac{1}{2x_0} + \frac{x_0}{2(x_0^2+y_0^2)}}_{\text{from z-quadrupole}} \underbrace{+\frac{2y_0}{(y_0-x_0)^2+y_0^2} - \frac{2y_0}{(y_0+x_0)^2+y_0^2}}_{\text{from x-quadrupole}}. \tag{2}$$

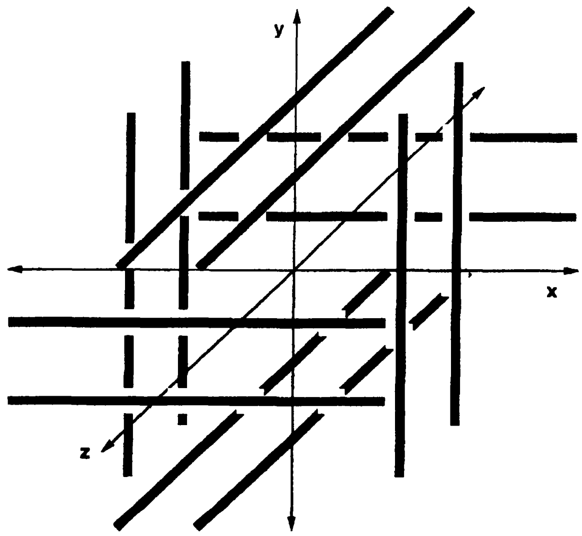

Figure 1: The filament dodecapole

Approximately the above equations provide an autonomous dynamical system for the motion of $x_0(t)$ and $y_0(t)$, and hereafter we shall concentrate on these equations.

To seek the similarity solutions for (1) and (2), we assume that all the variables develop with the same time dependence, $f(t)$, and substitute

$$x_0(t) = f(t)\xi, \qquad y_0(t) = f(t)\eta. \tag{3}$$

into the equations [5]. Then we obtain

$$f\dot{f}\,\xi = \frac{1}{2\eta} - \frac{\eta}{2(\xi^2+\eta^2)} - \frac{2\xi}{(\eta-\xi)^2+\xi^2} + \frac{2\xi}{(\eta+\xi)^2+\xi^2} \tag{4}$$

$$f\dot{f}\,\eta = -\frac{1}{2\xi} + \frac{\xi}{2(\xi^2+\eta^2)} + \frac{2\eta}{(\eta-\xi)^2+\eta^2} - \frac{2\eta}{(\eta+\xi)^2+\eta^2}. \tag{5}$$

Next we separate variables into the time and space parts by setting

$$f\dot{f} = c \tag{6}$$

where c is a real constant determined by ξ and η which satisfy

$$\begin{aligned} &\frac{1}{\xi}\left[\frac{1}{2\eta} - \frac{\eta}{2(\xi^2+\eta^2)} - \frac{2\xi}{(\eta-\xi)^2+\xi^2} + \frac{2\xi}{(\eta+\xi)^2+\xi^2}\right] \\ = \; &\frac{1}{\eta}\left[-\frac{1}{2\xi} + \frac{\xi}{2(\xi^2+\eta^2)} + \frac{2\eta}{(\eta-\xi)^2+\eta^2} - \frac{2\eta}{(\eta+\xi)^2+\eta^2}\right]. \end{aligned} \tag{7}$$

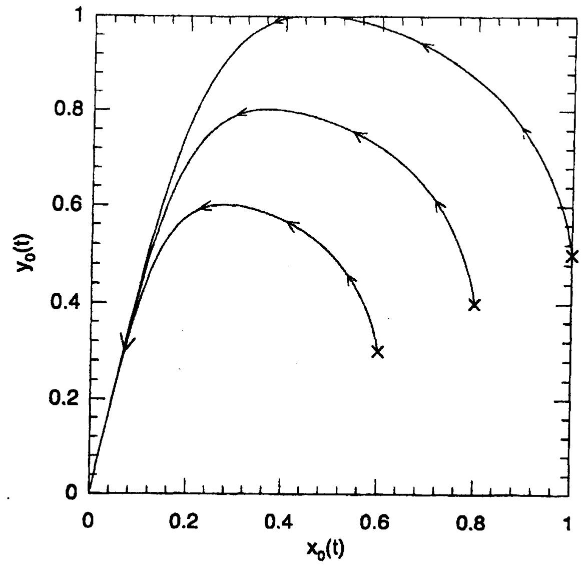

Figure 2: Trajectories of (1) and (2) with different initial conditions.

Letting $r = \eta/\xi$, (7) reduces to $4r^8 - 80r^6 + 17r^4 - 80r^2 + 4 = 0$ which has the solutions

$$\begin{cases} r_{1,2}^2 = s_+ \pm \sqrt{s_+^2 - 1} \\ r_{3,4}^2 = s_- \pm \sqrt{s_-^2 - 1} \end{cases} \tag{8}$$

where $s_\pm = 5 \pm \frac{\sqrt{391}}{4}$. The real positive solutions are $r = 4.4538127166...$ and the inverse.

The time dependence can be found by solving (6) with an initial condition $f(0) = 1$, and we find

$$f = \sqrt{2ct+1} = \sqrt{2c(t - t_{crit})} \tag{9}$$

where $t_{crit} = -\frac{1}{2c}$. According to the sign of c the system either contracts (if $c < 0$) or expands (if $c > 0$), respectively. Using r and ξ the constant c has the following form

$$c = \frac{4 - 16r^2 + 15r^4}{2\xi^2 r(r^2+1)(r^4+4)}. \tag{10}$$

We can see from the analysis above that the trajectory of x_0, y_0 is a straight line of slope r which crosses the origin at $t = t_{crit}$, and the critical time is a function of the initial position and r only.

A linear stability analysis of the similarity solution can be conducted by introducing a moving coordinates with the similarity solution and a new time variable. [6] The detail would be presented elsewhere.

Figure is a plot of trajectories of (1) and (2) with different initial conditions marked by X. We see that each trajectory approaches to a line with a slope of r obtained by the above discussion and then goes to the origin. We have verified that the calculated collapse time from (10) provides a satisfactory estimate once the trajectory is close to the straight line.

The last time we met was on July 16th in Kyoto. After discussion about extension of the subject in this article at RIMS, we went out to the "Yoiyama" of the Gion festival. The noise and crowd were overwhelming and bewildering, but we strolled around energetically. At one spot on a small street we stopped and stayed rather long without a word, where a large group of people in the festival costume were playing the Gion festival music endlessly. Even now, that sound of music, slow and monotone with flutes, bells and drums, remains in my head as a requiem for Rich.

References

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